of Eqs. (7) an equation for η_{sh} is obtained as:

$$\eta_{sh}^{1/2}\left\{\frac{3(\gamma+1)M^4\delta^2}{2^{3/2}\beta^{3/2}} - \eta_{sh}^{1/2}\left[1+\frac{(\gamma+1)M^4\delta^2}{2\beta^2}\right]\right\} = 0 \quad (8)$$

Disregarding the trivial solution gives

$$\eta_{sh} = \frac{9(\gamma+1)^2M^8\delta^4}{8\beta^3}\Big/\left[1+\frac{(\gamma+1)M^4\delta^2}{2\beta^2}\right]^2 \quad (9)$$

From Eqs. (9) and (4) the pressure rise across the shock wave is determined as

$$\frac{\Delta p}{p_\infty} = -\gamma M^2\frac{u}{V} = \frac{(\frac{3}{2})[\gamma(\gamma+1)M^6\delta^4/\beta^2]}{1+[(\gamma+1)M^4\delta^2/2\beta^2]} \quad (10)$$

and the angle that the conical shock wave makes with the linearized Mach cone is

$$\epsilon = \frac{(\frac{3}{8})[(\gamma+1)^2M^6\delta^4/\beta^3]}{1+[(\gamma+1)M^4\delta^2/2\beta^2]} \quad (11)$$

Equations (10) and (11) differ from the results of Lighthill and Whitham only by the appearance of the additional $(\gamma+1)M^4\delta^2/2\beta^2$ factor in the denominators. The validity of these equations easily can be checked by comparing results given by them with those obtained by numerically integrating the exact adiabatic equations of motion.[3] The angular difference (ϵ) between the shock cone and the undisturbed Mach cone is shown in Fig. 1 for two cone semiangles, 7.5° and 10°, as a function of Mach number. Results given by Eq. (11) are compared with those obtained from Whitham's formula[2] and from cone tables.[4] For the conditions illustrated, results from the present study show significantly better agreement with results from the cone tables, especially at the higher Mach numbers. As mentioned previously, any greater accuracy would have to be obtained by resorting to a uniformly valid second-order theory.

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Stress Concentration in a Cylindrical Shell with an Elliptical Cutout

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Nomenclature

$a^{\alpha\beta}$	= metric tensor
b	= half the focal width of the ellipse
B	= bending moment at the boundary
c	= half the major axis of the ellipse
d	= axis ratio for the ellipse
D_i	= arbitrary constant
$d_{\alpha\beta}$	= curvature of the shell
E	= Young's modulus
$e^{\alpha\beta}$	= alternating tensor
F	= auxiliary function
$G(x^1)$ and $H(x^2)$	= factors of the auxiliary function F
h	= shell thickness
N	= component of boundary force along the boundary normal
n^α	= boundary normal
p	= internal pressure
P_n, P_t, P_s, and P_b	= factors appearing in the components of the boundary forces
R	= cylinder radius
S	= component of boundary force along the shell normal
s	= a hole parameter
T	= component of boundary force along the boundary tangent
t^α	= boundary tangent
w	= deflection along the shell normal
(x,y)	= generator coordinate system
(x^1,x^2)	= ellipse system
Z^{I},Z^{II}	= Bessel function, first and second kind
$\mathrm{ceh}_i(x^2)$, $\mathrm{seh}_i(x^2)$	= periodic Mathieu functions
α_{membr}, α_{tot}	= stress concentration factors
ϕ	= complex stress function
ψ	= Airy's stress function
$\psi\vert_\alpha$	= covariant derivative of ψ
$\Delta\psi$	= first invariant of the second derivatives of ψ
ν	= Poisson's ratio

Subscripts

$,x,\alpha$	= partial derivative
unif	= uniform axial tension
membr	= membrane
tot	= total
pl	= plane

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Introduction

Although many papers have been published on the subject of stress concentration at circular and nearly circular holes in cylindrical shells, it appears that the only two that treat elliptical holes are those by Venkitapathy[2] and Murthy.[3] In both cases the major axis of the ellipse is oriented along a generator to the cylinder. In the paper by Venkitapathy the solution of the differential equation is incomplete; compare Eq. (2.17) of Ref. 1 with Eq. (10) of the present Note. Murthy uses a perturbation method, the parameter of which describes the size of the hole, and in the paper he states that the solution applies only to small holes.

The present Note also deals with a circular cylindrical shell weakened by an elliptical hole, but here it is the minor axis that is oriented along a generator to the cylinder. The elastic stress concentration due to axial tension is examined. As the analysis is based on shallow shell theory, the Note begins with an outline of that theory. The general solution of the differential equation is found after a simplifying variable substitution. The boundary conditions, treated with the help of Fourier analysis, together with assumptions regarding the symmetry and asymptotic behavior of the solution, then determine a unique stress field. This method can in certain cases be applied with confidence to holes up to

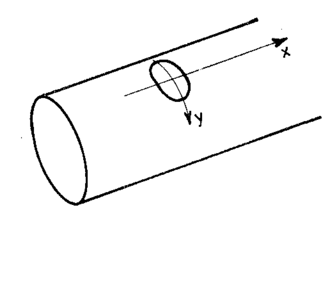

Fig. 1 Cylindrical shell with elliptical cutout and generator coordinate system (x, y).

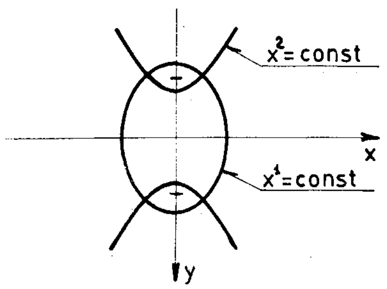

Fig. 2 Ellipse coordinate system (x^1,x^2) consisting ellipses and hyperbolas confocal with the cutout.

about 6 times the size of those treated by Murthy. The axis ratio lies between 0.25 and 0.90. The results show an increase in the membrane stresses of up to 60% as compared with the stresses in a plane disc with a hole of the same eccentricity. In addition, we find considerable bending stresses.

While this Note was being completed, Murthy and Rao published a paper[4] about an elliptical hole oriented as in the present Note. As the perturbation method of Ref. 3 is used in Ref. 4 too, the results are valid only for small holes.

Fundamental Equations

Our approach to the elastic stresses in a shell is the application of shallow shell theory. This can be formulated in terms of two dependent variables, namely w, the deflection along a normal to the shell and ψ, an Airy's stress function. If E is Young's modulus and ν is Poisson's ratio for the material, the governing equations are:

$$[Eh^3/2(1 - \nu^2)]\Delta\Delta w - e^{\alpha\gamma}e^{\beta\delta}\psi|_{\gamma\delta}d_{\alpha\beta} - p = 0 \quad (1)$$

$$(1/Eh)\Delta\Delta\psi + e^{\alpha\gamma}e^{\beta\delta}w|_{\gamma\delta}d_{\alpha\beta} = 0 \quad (2)$$

Here h is the shell thickness, p the internal pressure and $d_{\alpha\beta}$ the shell curvature. The alternating tensor is $e^{\alpha\beta}$, and the covariant derivative of ψ is written as $\psi|_\alpha$. The symbol $\Delta\psi$ means $\psi|_\alpha{}^\alpha$.

The boundary has the normal n^α and the tangent t^α. If it is subjected to a bending moment B and a force whose components along the boundary normal, boundary tangent and shell normal are N, T, and S respectively, and if s denotes the curve length and $a^{\alpha\beta}$ the metric tensor of the coordinate system, then the boundary conditions are:

$$N = \psi|_{\alpha\beta}t^\alpha t^\beta \quad (3)$$

$$T = -\psi|_{\alpha\beta}n^\alpha t^\beta \quad (4)$$

$$S = [Eh^3/12(1 - \nu^2)](-w|_\beta{}^{\alpha\beta}n_\alpha - \partial/\partial s\{[(1 - \nu)w|^{\alpha\beta} + \nu w|_\gamma{}^\gamma a^{\alpha\beta}]n_\alpha t_\beta\}) \quad (5)$$

$$B = [Eh^3/12(1 - \nu^2)]\{[(1 - \nu)w|^{\alpha\beta} + \nu w|_\gamma{}^\gamma a^{\alpha\beta}]n_\alpha n_\beta\} \quad (6)$$

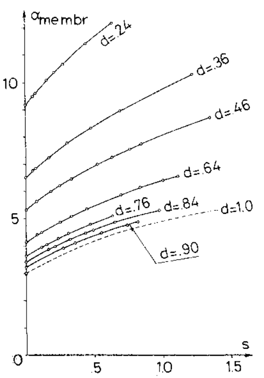

Fig. 3 Membrane stress concentration factor α_{membr} as a function of the sizeparameter s and the axis ratio d of the cutout. The parameter s is defined by $s = c^2[12(1 - \nu^2)]^{1/2}/(16Rh)$.

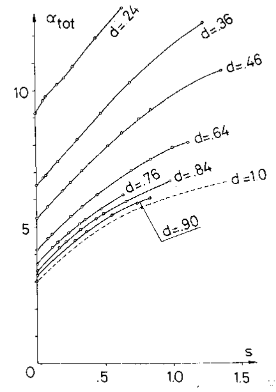

Fig. 4 Total stress concentration factor α_{tot} as a function of sizeparameter s and axis ratio d of the cutout.

We use two coordinate systems to describe the cylindrical shell with the elliptical hole. The generator system (x,y), shown in Fig. 1, has its x-axis along a generator to the cylinder. The ellipse system (x^1,x^2), shown in Fig. 2, consists of confocal ellipses and hyperbolas. In the latter system, the hole boundary is a coordinate curve. The coordinates are related by

$$x = b \sinh x^1 \sin x^2; \quad y = b \cosh x^1 \cos x^2$$

where b is half the focal width of the ellipse.

In the case of a circular cylindrical shell it is possible to combine the two dependent variables in the differential Eqs. (1) and (2) into one complex variable, ϕ, defined by:

$$\phi = \psi + [Eh^3/12(1 - \nu^2)]wi \quad (7)$$

This enables us to replace Eqs. (1) and (2) by a single equation, which in the generator system is:

$$\Delta\Delta\phi - i\{[12(1 - \nu^2)]^{1/2}/Rh\}\phi_{,xx} - ip = 0 \quad (8)$$

where R is the radius of the cylinder and $\phi_{,xx}$ is the second partial derivative of ϕ with respect to x.

The boundary conditions Eqs. (3–6) expressed in the ellipse system are:

$$N = \{2/b^2(\cosh 2x^1 - \cos 2x^2)^2\}Re[\phi_{,22}(\cosh 2x^1 - \cos 2x^2) + \phi_{,1}\sinh 2x^1 - \phi_{,2}\sin 2x^2] \quad (9)$$

with a similar expression for the force component T. The two remaining conditions contain combinations of the partial derivatives of $Im(\phi)$ up to and including the third order. For the real part of the parentheses [ ] in Eq. (9) we introduce the symbol P_n. The analogous factors in the other components are P_t, P_s, and P_b.

Solution of the Differential Equation

Lekkerkerker has shown in Ref. 1 that the substitution

$$\phi = F_c \cosh(Kx) + F_s \sinh(Kx) \quad (10)$$

transforms the governing differential Eq. (8) into a membrane wave equation in the auxiliary functions F, provided that the constant K is chosen properly. To solve this equation in the ellipse system, we can separate variables in F:

$$F(x^1,x^2) = G(x^1)H(x^2)$$

Now it turns out that $H(x^2)$ and $G(x^1)$ must be solutions of the ordinary and modified Mathieu equations, respectively. In addition, we require that the auxiliary functions F (and hence the stress function ϕ) are periodic in x^2 with the period 2π, and that they have properties that ensure the symmetry of the stress function ϕ via the substitution Eq. (10). The following series expansions are suitable for this purpose:

$$F_c(x^1,x^2) = \sum_{j=0}^{\infty} D_{2j}\,\mathrm{ceh}_{2j}(x^2)G_{2j}(x^1) \quad (11)$$

$$F_s(x^1,x^2) = \sum_{j=0}^{\infty} D_{(2j+1)}\,\text{seh}_{(2j+1)}(x^2)G_{(2j+1)}(x^1) \qquad (12)$$

Here $\text{ceh}_{2j}(x^2)$ and $\text{seh}_{(2j+1)}(x^2)$ denote the periodic ordinary Mathieu functions similar to $\cos(2jx^2)$ and $\sin[(2j + 1)x^2]$, respectively, in the sense that they approach these trigonometric functions when the size of the hole tends to zero. The corresponding modified Mathieu functions are $G_{2j}(x^1)$ and $G_{(2j+1)}(x^1)$ respectively. The constants D_i are arbitrary.

In each term of the series on the right-hand sides of Eqs. (11) and (12) we determine the periodic Mathieu functions by calculating their Fourier coefficients by a method described by Abramowitz in Ref. 5, with linear equations for the coefficients and a power series for a characteristic value. The modified Mathieu functions are described by a similar but more complicated expansion involving the same coefficients. In order to obtain a satisfactory rate of convergence, we have chosen the Bessel function product series (see Ref. 5, formulas 20.6.7 and 20.6.10) to compute the value of the modified functions $G_i(x^1)$ for the value of x^1 that corresponds to the boundary of the hole.

By inserting the series for the auxiliary functions F [Eqs. (11) and (12)] in the substitution Eq. (10) we obtain the trigonometric series for the stress function ϕ along the boundary of the hole. It too, contains the arbitrary constants D_i. Formula (10) also gives us the trigonometric series for the partial derivatives of ϕ; these are needed in the 4 boundary force equations, Eq. (9). The right-hand side of Eq. (9) is thus represented by the trigonometric series for the factor P_n, the coefficients of which still contain the arbitrary constants D_i. The same applies to the factors P_t, P_s and P_b.

Boundary Conditions

The physical boundary conditions of the problem are: a) far from the hole there is uniform axial tension; and b) the boundary of the hole must be free of stresses. For the sake of convenience we use superposition. To a uniformly distributed axial stress we add a stress field that neutralizes the uniform stress at the hole. The neutralizing field must then fulfil the following conditions: A) it must vanish far from the hole; and B) it must cancel the uniform stress at the hole boundary.

Condition (A) uniquely determines the modified Mathieu functions $G_i(x^1)$. The present choice of ϕ [Eq. (7)] leads to a value of the parameter of the modified Mathieu equation, for which only the fourth fundamental solution converges to zero far from the hole. The fourth solution is the one containing $Z^{\text{I}} - iZ^{\text{II}}$, Z^{I}, and Z^{II} being Bessel functions of the first and second kind.

Condition (B) is used in the following way: from the defining Eq. (7) and the force Eqs. (3–6) we find the stress function ϕ_{unif} corresponding to the uniform axial tension N_{unif}. When this stress function is inserted in the force equations as formulated in Eq. (9), we obtain the boundary forces and moments that are to be neutralized. We find that the boundary conditions for the neutralizing stress field are:

$$N = [-N_{\text{unif}}/2(\cosh 2x^1 - \cos 2x^2)^2]\{\tfrac{1}{2}(1 + \cosh 2x^1) \times \cos 4x^2 + [-(\cosh 2x^1)^2 - 2\cosh 2x^1 - 1]\cos 2x^2 + \tfrac{3}{2}(1 + \cosh 2x^1) + (\sinh 2x^1)^2\} \qquad (13)$$

with a similar equation for T, and

$$S = B = 0 \qquad (14)$$

In the previous section it is described how we can find the trigonometric series for the factors P_n, P_t, P_s, and P_b in the boundary forces corresponding to the solution [Eqs. (10–12)] of the differential equation. These Fourier coefficients contain the arbitrary constants D_i from the series on the right-hand side of Eqs. (11) and (12). By equating these coefficients to those of the trigonometric series in the parentheses { } in the neutralizing conditions Eqs. (13) and (14), we obtain a system of linear equations that determine the constants D_i.

Because of the hole, the shell is not a simply connected region, and we have to impose additional boundary conditions. They must insure that it is possible to integrate the strains at the hole to single-valued deformations and slopes of deformations. Lekkerkerker has shown in Ref. 1 that for our choice of the stress function ϕ [Eq. (7)] and the symmetry of the present problem, these conditions are automatically satisfied.

Results

The previous section shows how it is possible to make a stress field that neutralizes the boundary forces due to uniform axial tension. When we now use superposition, the final result appears. The membrane forces and the bending moment at the boundary of the hole are especially interesting. The formulas giving the forces and the moments on cuts perpendicular to the boundary are more or less similar to Eq. (9), and the computations are similar to those described at the end of the section "Solution of the Differential Equation," with the exception that the constants D_i are known at the present stage.

Figs. 3 and 4 show the membrane stress concentration factor α_{membr} and the total stress concentration factor α_{tot}. They are defined by:

$$\alpha_{\text{membr}} = \frac{\text{Max}(N)}{N_{\text{unif}}} \qquad \text{and} \qquad \alpha_{\text{tot}} = \frac{\text{Max}(\sigma)}{\sigma_{\text{unif}}}$$

where σ_{unif} is the uniform axial stress. At the ends of the major axis of the hole the membrane force is $\text{Max}(N)$ and the maximum stress is $\text{Max}(\sigma)$.

The parameters in Figs. 3 and 4 are d, the axis ratio of the ellipse, and s, a dimensionless parameter. The latter is defined by $s = c^2[12(1 - \nu^2)]^{1/2}/(16Rh)$ where c is half the major axis of the hole. The small circles on the curves show the results of the computer runs for which the differential equations and boundary conditions were sufficiently well satisfied.

The dotted curves with $d = 1.0$ enable us to compare the present results with the stress concentration at a circular hole in a cylindrical shell as treated by Lekkerkerker in Ref. 1.

We may consider for example a shell with radius $R = 300$ mm, thickness $h = 1$ mm, Poisson's ratio $\nu = 0.3$, and with a hole, the major and minor axes of which are 37 mm and 17 mm. The parameters for the curves are then $d = 0.46$ and $s = 0.94$, and Figs. 3 and 4 indicate that $\alpha_{\text{membr}} = 7.9$ and $\alpha_{\text{tot}} = 9.6$. In a plane disk the stress concentration at an elliptical hole is $\alpha_{pl} = 1 + 2/d$, and for our hole we find $\alpha_{pl} = 5.3$.

This shows that the curvature of the shell has a considerable influence on the stress concentration at an elliptical hole with the minor axis along a generator to the shell, when the loading is axial tension.

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